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Stability In a Fractional Order Three Species Interactions Model

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ABSTRACT:
The dynamical behavior of a fractional order prey - predator model is investigated in this paper. The equilibrium points are computed and stability of the equilibrium points are analyzed. The phase portraits are obtained for different sets of parameter values. Numerical simulations are performed and they exhibit rich dynamics of the fractional model.2010 Mathematics Subject Classification. 39A30, 92D25, 92D40.

KEY WORDS AND PHRASES: Fractional Order, differential equations, Global analysis, Prey - Predator, stability.

INTRODUCTION:
The theory of fractional calculus goes back to Leibniz's note in his letter to Hospital, dated 30 September 1695, in which the meaning of the derivative of order [1,2]is discussed. In recent years, there has been a great deal of interest in fractional differential equations. During the last decade fractional calculus has been applied to almost every field of science, engineering, and mathematics. Fractional differential equations (FDEs) have found applications in many problems in physics and engineering. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. Recently, fractional calculus was introduced to the stability analysis of nonlinear systems. Analysis of stability is fundamental to any control system. However, the work on the topic of stability for fractional order predator-prey system is rare.

1. FRACTIONAL DERIVATIVES AND INTEGRALS:

Definition 1. The fractional integral (or the Riemann-Liouville integral) of order $\beta \in \mathbb{R}^+$ for the function $f(t)\ t>0\ (f:\mathbb{R}^+ \to \mathbb{R})$ is defined by

$$I_\beta^\alpha f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s)\,ds, \quad t>0 \qquad (1)$$

The fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$ is defined by two (nonequivalent) ways:

(i). Riemann-Liouville fractional derivative: take fractional integral of order $(n-\alpha)$ and then take nth derivative as follows:

$$D_*^\alpha f(t) = D_*^n I_a^{n-\alpha} f(t),\ D_*^\alpha = \frac{d^n}{dt^n},\ n = 1, 2, \ldots \qquad (2)$$

(ii). Caputo-fractional derivative take nth derivative, and then take a fractional integer of order $(n-\alpha)$

$$D^\alpha f(t) = I_a^{n-\alpha} D_*^n f(t),\ n = 1, 2, \ldots \qquad (3)$$

1.1. **SOME PROPERTIES OF FRACTIONAL DERIVATIVES AND INTEGRALS:**

The main properties of fractional derivatives/integrals are as follows (Oldham and Spanier, 1974):[1,2]

1. If $f(t)$ is an analytical function of t, then its fractional derivative ${}_aD_\alpha^t f(t)$ is an analytical function of t , α .
2. For $\alpha = n$, where n is integer, the operation ${}_0D_t^\alpha$ gives the same result as classical differentiation of integer order n.
3. For $\alpha = 0$ the operation ${}_aD_\alpha^t f(t)$ is the identity operator:
   ${}_0D_t^\alpha f(t) = f(t)$.
4. Fractional differentiation and fractional integration are linear operations:
   ${}_aD_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda\, {}_aD_t^\alpha f(t) + \mu\, {}_aD_t^\alpha g(t)$
5. The additive index law (semi group property)
   ${}_0D_t^\alpha\, {}_0D_t^\beta f(t) = {}_0D_t^\beta\, {}_0D_t^\alpha f(t) = {}_0D_t^{\alpha+\beta}$.

holds under some reasonable constraints on the function $f(t)$. The fractional-order derivative commutes with integer-order derivative

$$\frac{d^n}{dt^n}\left(aD_t^r\ f(t)\right) = {}_aD_t^r\ f(t)\left(\frac{d^n\ f(t)}{dt^n}\right)$$

under the condition $t = a$ we have $f^{(k)}(a) = 0$, $(k = 0,1,2,\ldots,n-1)$. The relationship above says the operators $\frac{d^n}{dt^n}$ and ${}_aD_t^r$ commute.

2. SOME LEMMAS

Lemma 1.[1] The following linear commensurate fractional - order autonomous system

$$D^\alpha x = Ax, \quad x(0) = x_0$$

is asymptotically stable if and only if $|arg\lambda| > \alpha\frac{\pi}{2}$ is satisfied for all eigenvalues (λ) of matrix A . Also, this system is stable if and only if $|arg\lambda| > \alpha\frac{\pi}{2}$ is satisfied for all eigenvalues (λ) of matrix A and those critical eigenvalues which satisfy $|arg\lambda| = \alpha\frac{\pi}{2}$ have geometric multiplicity one, where $0 < \alpha < 1, x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.

Lemma 2. [1]Consider the following autonomous system for internal stability definition

$$D^\alpha x(t) = Ax(t), \quad x(0) = x_0$$

with $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]^T$ and its n-dimensional representation:

$$\begin{aligned} D^{\alpha_1}x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ D^{\alpha_2}x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ &\cdots\cdots \\ D^{\alpha_n}x_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{aligned} \tag{4}$$

where all α_i 's are rational numbers between 0 and 2. Assume m to be the LCM of the denominators u_i 's of α_i 's, where $\alpha_i = \frac{u_i}{v_i}, u_i, v_i \in \square^{+}$ for $i = 1, 2, ..., n$ and we set $\gamma = \frac{1}{m}$. Define:

$$det\begin{bmatrix} \lambda^{m\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{m\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{m\alpha_n} - a_{nn} \end{bmatrix} = 0 \qquad (5)$$

The characteristic equation (7) can be transformed to integer - order polynomial equation if all α_i 's are rational number. Then the zero solution of system (6) is globally asymptotically stable if all roots λ_i 's of the characteristic (polynomial) equation (7) satisfy:

$$|arg(\lambda_i)| > \gamma \frac{\pi}{2} \forall i.$$

3. MODEL DESCRIPTION AND EXISTENCE OF EQUILIBRIUM POINTS:

The interaction between the predator and prey has attracted a lot of attention and many good results have already been reported. In 1926 Volterra came up with a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. They were proposed independently by Alfred J.Lotka in 1925 [1]. The equations are

$$N' = N(a - bP) \; ; \; P' = P(cN - d)$$

where a, b, c and d are positive constants. Recent years have witnessed rapid development in the field of application of fractional calculus in biology, economics and engineering [4] Several authors formulated fractional order systems and analyzed the dynamical and qualitative behavior of the systems [3, 5, 6, 7]. Following this trend, in this paper, we propose a system of fractional order prey-predator model. The stability of equilibrium points is studied. Numerical solutions and simulations of this model are provided. We consider the fractional order of the model [5]in as follows:

$$D^{\alpha_1}x(t) = ax(t) - bx(t)z(t)$$
$$D^{\alpha_2}y(t) = ry(t)[1 - y(t)] - cy(t)z(t) \qquad (5)$$
$$D^{\alpha_3}z(t) = ex(t)z(t) + fy(t)z(t) - dz(t)$$

where the parameters $r, a, b, c, d, e, f > 0$ and $\alpha_1, \alpha_2, \alpha_3$ are fractional orders. To evaluate the equilibrium points, let us consider

$$D^{\alpha_1}x(t) = 0; D^{\alpha_2}y(t) = 0; D^{\alpha_3}z(t) = 0$$

The fractional order system has five equilibria $E_0 = (0,0,0)$ (trivial), $E_2 = (0,1,0)$ (axial), $E_2 = \left(\frac{d}{e}, 0, \frac{a}{b}\right)$ (axial), $E_3 = \left(0, \frac{d}{f}, \frac{r(f-d)}{cf}\right)$ (axial) and $E_4 = \left(\frac{acf}{ber} + \frac{1}{e}(d-f), 1 - \frac{ac}{df}, \frac{a}{b}\right)$ (Interior).

To accommodate biological meaning, the existence condition for the equilibria require that they are nonnegative. It obvious E_0, E_1 and E_2 always exist, E_3 exist when $f > d$. The interior equilibrium E_4 exist when $acf > br(f-d)$ and $br > ac$.

4. LOCAL STABILITY OF FIXED POINTS:

Based on (6), to investigate the local stability of each fixed point (x^*, y^*, z^*) we provide the Jacobian matrix J

$$J(x,y,z)=\begin{bmatrix} a-bz & 0 & -bx \\ 0 & r(1-2y)-cz & -cy \\ ez & fz & ex+fy-d \end{bmatrix} \quad (7)$$

For E_0, we have

$$J(E_0)=\begin{bmatrix} a & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & -d \end{bmatrix}$$

The eigenvalues are $\lambda_1 = a, \lambda_2 = r$ and $\lambda_3 = -d$. It is clear that E_0 is saddle point, while for E_1 we have

$$J(E_1)=\begin{bmatrix} a & 0 & 0 \\ 0 & -r & -c \\ 0 & 0 & f-d \end{bmatrix}$$

The eigenvalues are $\lambda_1 = a, \lambda_2 = -r$ and $\lambda_3 = f-d$, E_1 is asymptotically stable when $f < 1+d$. Jacobian of E_2 is

$$J(E_2)=\begin{bmatrix} 0 & 0 & -\frac{bd}{e} \\ 0 & r-\frac{ac}{b} & 0 \\ \frac{ae}{b} & \frac{af}{b} & 0 \end{bmatrix}$$

which has the following eigenvalues: $\lambda_1 = r-\frac{ac}{b}$ and $\lambda_{2,3} = \pm\sqrt{ad}$. Since both of λ_2 and λ_3 are negative, local stability of E_2 is determined by λ_1. Hence E_2 is stable note when $b(r-1) < ac$.Local stability of $E_3 = \left(0, \frac{d}{f}, \frac{r(f-d)}{cf}\right)$ is determined by investigating the eigenvalues of

$$J(E_3)=\begin{bmatrix} a-\frac{br(f-d)}{cf} & 0 & 0 \\ 0 & -\frac{dr}{f} & -\frac{dc}{f} \\ \frac{re(f-d)}{cf} & \frac{r(f-d)}{b} & 0 \end{bmatrix},$$

namely

$$\lambda_1 = a+\frac{br(d-f)}{cf},\ \lambda_{2,3} = -\frac{dr}{2f} \pm \frac{1}{2f}\sqrt{dr[4f(d-f)+dr]}.$$

obvious E_3 is stable when $\frac{br(f-d)}{f} < (1-a)c$. Finally the local stability of the interior equilibrium point is investigated by considering the Jacobian matrix

$$J(E_4)=\begin{bmatrix}0 & 0 & A\\ 0 & B & C\\ D & E & 0\end{bmatrix},$$

Where $A=\frac{b}{e}(f-d)-\frac{acf}{er}$, $B=-r-\frac{3ac}{b}$, $C=\frac{ac^2}{br}-c$, $D=\frac{ae}{b}, E=\frac{af}{b}$. The characteristic polynomial $P(\lambda)$ for E_4 is

$$P(\lambda)=\lambda^3-a_1\lambda^2+a_2\lambda-a_3$$

Where

$$a_1=B, a_2=-EC-AD, a_3=-ABD$$

It is obvious that $a_1 > 0$ and $a_3 > 0$. If $a_1a_2 > a_3$, then Routh Hurwitz criterion implies that all roots of $P(\lambda)$ have negative real parts, or in other words, E_4 is a stable point. It can be shown that equation

$$a_1a_2 - a_3 = -EBC$$

is positive if $ac > br$. These conditions are in contrast to the existence condition of E_4. It means that E_4 is unstable. This section is ended by summarizing the existence and stability condition of all the equilibrium point in the following table:

Equilibrium Point	**Existence Condition**	**Stability Condition**
E_0	-	Saddle
E_1	-	$f<1+d$
E_2	-	b(r -1) < ac
E_3	$f>1+d$	acf< br$(f-d)$
E_4	b(r-1) > ac and acf > br(f - d)	ac > b(1-r)

5. DYNAMIC BEHAVIOR WITH NUMERICAL SOLUTIONS:

Numerical solution of the fractional- order Prey- Predator system is given as follows [1]:

$${}_aD_t^q\, y(t)=f(y(t),t)$$

can be expressed as

$$y(t_k)=f\left(y(t_k),t_k\right)h^q-\sum_{j=v}^{k}c_j^{(q)}y(t_{k-j}).$$

The general numerical solution of the fractional differential equation [1]

$$x(t_k)=\left(ax(t_{k-1})-bx(t_{k-1})z(t_{k-1})\right)h^{\alpha_1}-\sum_{j=v}^{k}c_j^{(\alpha_1)}x(t_{k-j})$$

$$y(t_k)=\left(ty(t_{k-1})[1-y(t_{k-1})]-cy(t_{k-1})z(t_{k-1})\right)h^{\alpha_2}-\sum_{j=v}^{k}c_j^{(\alpha_2)}y(t_{k-j})$$

$$z(t_k)=\left(ex(t_{k-1})z(t_{k-1})+fy(t_{k-1})z(t_{k-1})-dz(t_{k-1})\right)h^{\alpha_3}-\sum_{j=v}^{k}c_j^{(\alpha_2)}z(t_{k-j})$$

where $T_{\{sim\}}$ is the simulation time, k = 1, 2, 3, ⋯ , N, for $N=\left[T_{\{sim\}}/h\right]$, and $\left(x(0),y(0),z(0)\right)$ is the initial conditions.

Example 1. Let us consider the parameters with values $r = 1$; $a = 13$; $b = 6$; $c = 4$; $d = 17$; $e = 20$; $f = 14$ and the derivative order $\alpha_1 = \alpha_2 = \alpha_3 = 0.96$ for these parameter the corresponding eigenvalues are $\lambda_1 = -38.5592$ and $\lambda_{2,3} = 5.7796 \pm i33.1406$ for E_4 which satisfy conditions $|arg\lambda| > \alpha\frac{\pi}{2}$. It means the system (6) is stable, see fig - 1. Also the characteristic equation of the linearized system (6) at the equilibrium point E_4 is

$$\lambda^{288} - 27\lambda^{192} + 685.99\lambda^{96} - 43637.64 = 0.$$

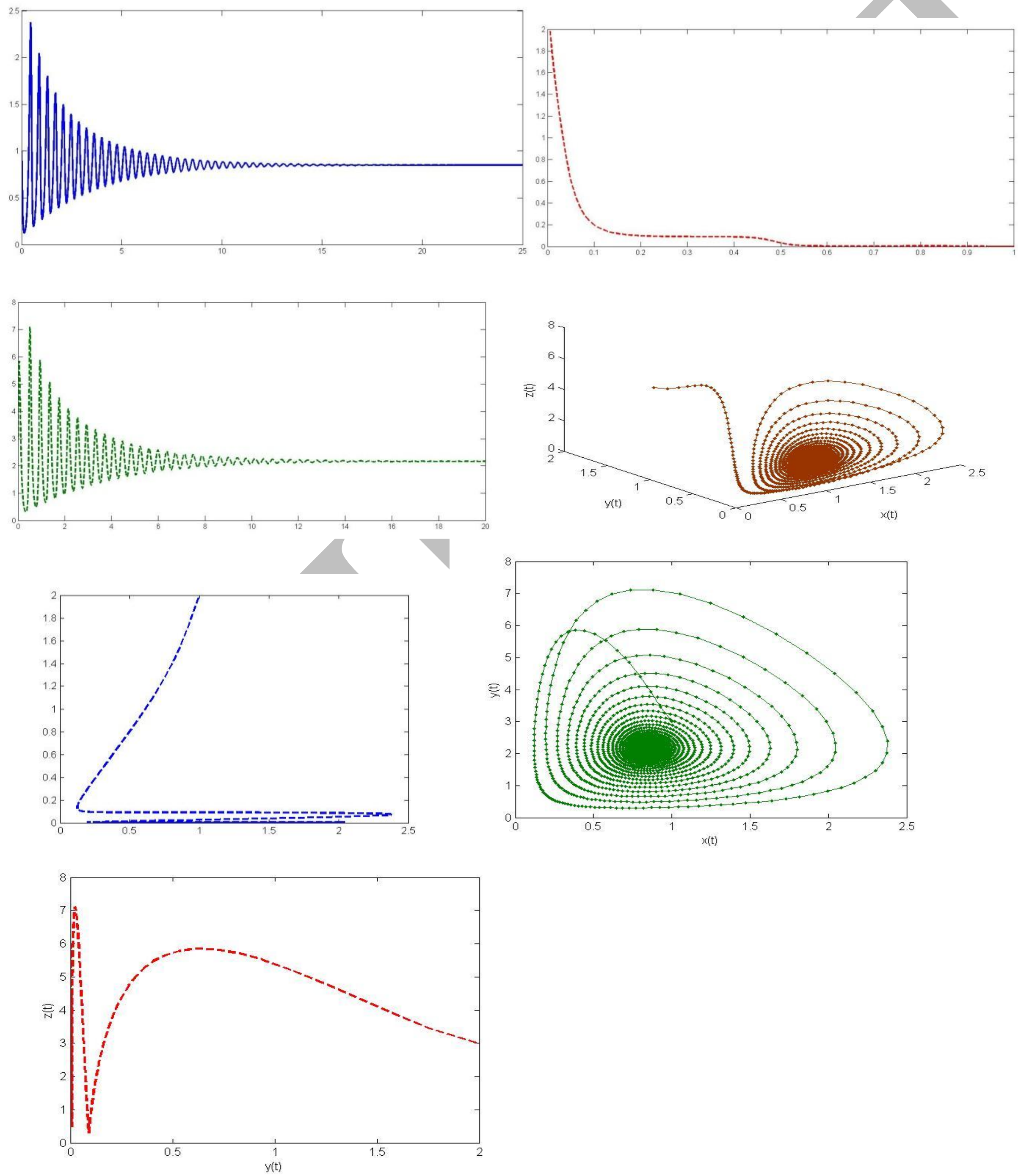

Figure 1. Time series and Phase diagram of fixed point E_4 with Stability

Example 2. Let us consider the parameters with values $r = 1; a = 6; b = 2; c = 3; d = 8; e = 3; f = 2,$ and the derivative order $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$ For these parameter the corresponding eigenvalues are $\lambda_1 = -31.9499$ and $\lambda_{2,3} = 1.9749 \pm i11.0588$ for E_4 , which is not satisfy conditions $|arg\lambda| > \alpha\frac{\pi}{2}$. It means the system (6) is Unstable, see fig - 2. Also The characteristic equation of the linearized system (6) at the equilibrium point E_4 is

$$\lambda^{297} - 28\lambda^{198} + 112\lambda^{99} - 4032 = 0.$$

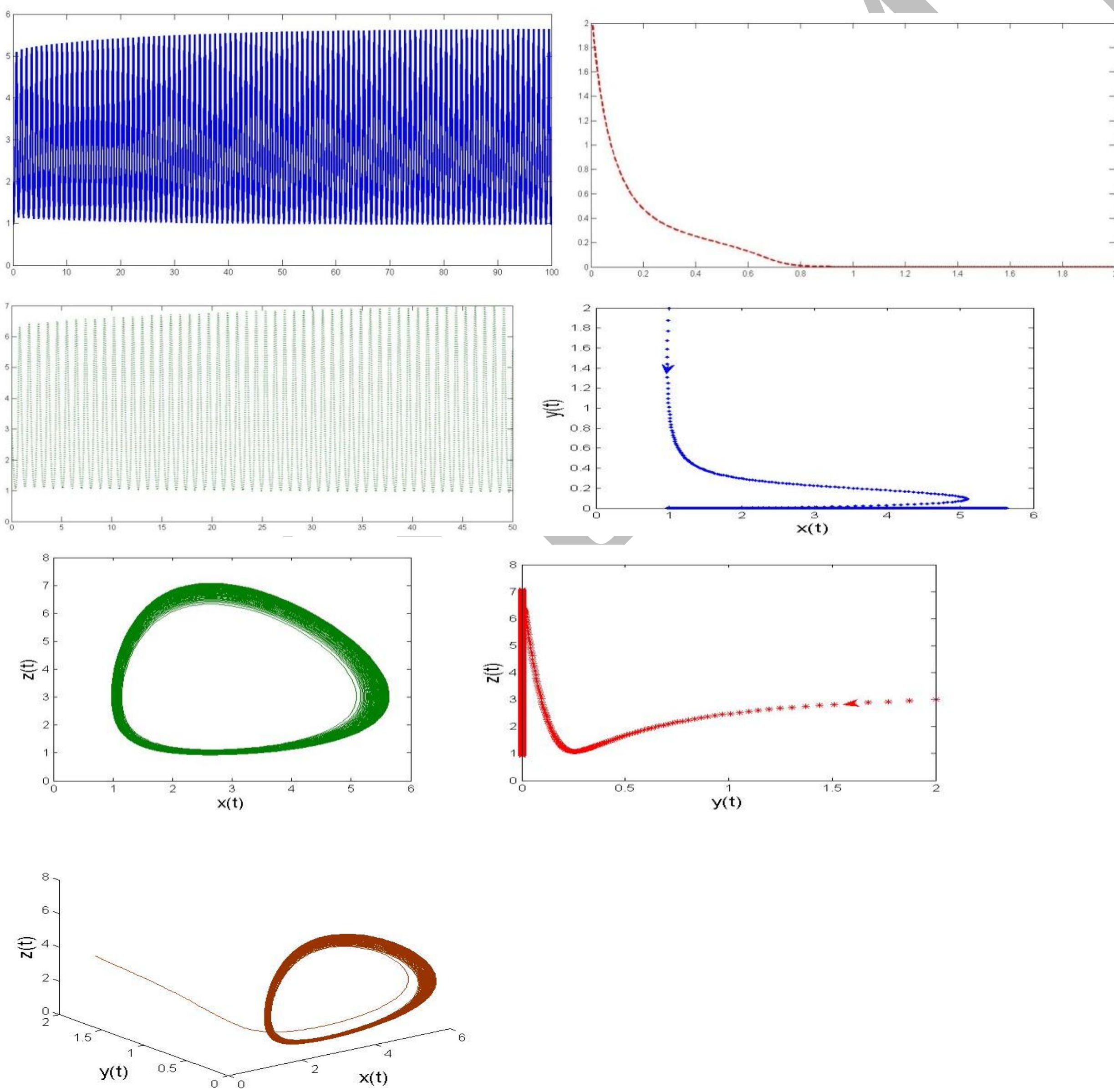

Figure 2. Time series and Phase diagram of fixed point E_4 with Unitability

In this paper, we have investigated the stability properties of a fractional order three species predator-prey model. The characteristic equation is introduced for the fractional order predator-prey system. Stability conditions for the fractional order predator-prey system are obtained. Illustrative examples are provided to support the theoretical analysis.

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