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Hydro magnetic Stability of Stratified Shear Flows in Sea Straits of Arbitrary Cross Section

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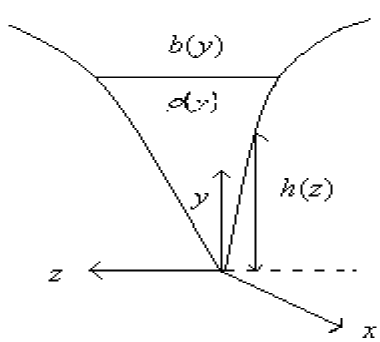
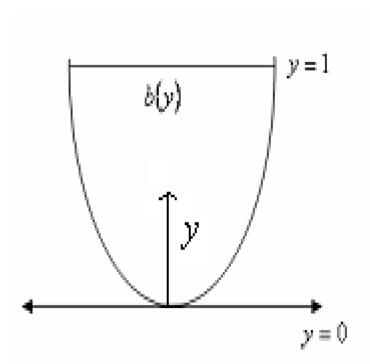
Abstract:
In this paper, the hydro magnetic stability of stratified shear flows in sea straits of arbitrary cross section has been discussed. Some results related to stability or instability have been obtained for oscillatory and non-oscillatory modes.

1 Introduction:

The stability of homogeneous shear flows and stratified shear flows of an inviscid fluid to infinitesimal disturbances has been extensively studied. **Drazin & Reid** (1981), **Yih** (1981), **Kochar & Jain** (1979), **Banerjee, et al.** (1988, 1994), **Padmini & Subbiah** (1993) and **Schmid & Henningson** (2001) are few names who worked in this direction. These studied are restricted to rectangular cross sections. However sea straits have rarely rectangular cross sections. For example, the Bab of Mandab has a deep central trough bounded by shallow flanges. **Pratt et al**.(2000) derived an extended version of Taylor-Goldstein equation for non-rectangular cross section. **Deng et al.** (2003) developed a more general theory for transversely uniform, time dependent, stratified flow in a channel of arbitrary cross section. For the extended Taylor-Goldstein problem, they showed that the familiar results for Taylor-Goldstein problem are unaffected by the geometry of cross section. **Hari Kishan & Neelu Chaudhary** (2006) discussed the hydro magnetic stability of stratified shear flows. **Subbiah & Ganesh** (2007) discussed the stability of homogeneous shear flows in sea straits of arbitrary cross sections. **Naresh Kumar, Hari Kishan and Ruchi Goel** (2011) discussed the hydro magnetic stability of stratified shear flows in presence of cross flow.

In this paper, the hydro magnetic stability of stratified shear flows in sea straits of arbitrary cross section has been discussed. The fluid flow considered here is shown in the following figures:





2 The Governing Stability Equation:

Let the waves be linearly propagating in a stratified background flow with velocity *U*(*y*), magnetic field *H* applied in *x*-direction and density $\rho_0(y)$. The channel is assumed in *x*-direction and the bottom elevation $h(z)$ has a single minimum with respect to cross channel coordinate *z*. The width of the channel at any elevation *y* is denoted by $b(y)$. If there are several minima of $h(z)$ then $b(y)$ represent the sum of widths of individual topographic troughs. Let *ρ, u,* (h_x, h_y, h_z) and *p* denote small perturbations from the density, *x* velocity, component of perturbations in magnetic field and hydrostatic pressure of the background flow, and let *w* and *v* denote the associated lateral and vertical velocity components. Employing the Bossiness approximation, the linearized, in viscid, hydrostatic equations of motion describing these fields are given by

$$\rho_0\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u+\frac{dU}{dy}v\right]=-\frac{\partial p}{\partial x}+\frac{\mu_e H}{4\pi}\frac{\partial h_x}{\partial x}, \qquad \ldots(1)$$

$$\rho_0\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)v=-\frac{\partial p}{\partial y}+\frac{\mu_e H}{4\pi}\frac{\partial h_y}{\partial y}-\rho g, \qquad \ldots(2)$$

$$\rho_0\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)w=-\frac{\partial p}{\partial z}+\frac{\mu_e H}{4\pi}\frac{\partial h_z}{\partial z}, \qquad \ldots(3)$$

$$\frac{\partial h_x}{\partial t}+U\frac{\partial h_x}{\partial x}=H\frac{\partial u}{\partial x}+h_y U', \qquad \ldots(4)$$

$$\frac{\partial h_y}{\partial t}+U\frac{\partial h_y}{\partial x}=H\frac{\partial v}{\partial x}, \qquad \ldots(5)$$

$$\frac{\partial h_z}{\partial t}+U\frac{\partial h_z}{\partial x}=H\frac{\partial w}{\partial x}, \qquad \ldots(6)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \qquad \ldots(7)$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0, \qquad \ldots(8)$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\rho+\frac{d\rho_0}{dy}v=0, \qquad \ldots(9)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \ldots(10)$$

We have to study the waves for which (h_x, h_y, h_z), *ρ, p, u, v* and *w* are uniform in *y*, implying that the isopycnal surfaces rise and fall uniformly across the channel. Such solutions are dynamically consistent only in the limit of long wave length compared to channel width. Integrating (5) across the channel at any *z* and applying the conditions *w* = *v*(*dh*/*dy*) at the two side walls leads to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+T(y)v=0, \qquad \ldots(11)$$

Where $T(y)=b^{-1}\frac{db}{dy}=\frac{d(\log b)}{dy}$.

Now applying the normal mode technique, i.e. the transformations

$$(u,v,h_x h_y h_z,p)=(\bar{u}(y),\bar{v}(y),\bar{h_x},\bar{h_y},\bar{h_z},\bar{p}(y))e^{i(kx-ct)}$$

in the equations (1), (2), (3), (4) and (11) and omitting the bar signs, we get

$$\rho_0(U-c)u+\frac{\rho_0}{ik}\frac{dU}{dy}v=-p+\frac{\mu_e H h_x}{4\pi}, \qquad \ldots(12)$$

$$\rho_0 ik(U-c)v=-\frac{dp}{dy}+\frac{\mu_e H}{4\pi}\frac{dh_y}{dy}-\rho g, \qquad \ldots(13)$$

$$\rho_0 ik(U-c)w=-\frac{\partial p}{\partial z}+\frac{\mu_e H h_z}{4\pi}, \qquad \ldots(14)$$

$$ik(U-c)h_x=ikHu+h_y U', \qquad \ldots(15)$$

$$ik(U-c)h_y=ikHv, \qquad \ldots(16)$$

$$ik(U-c)h_z=ikHw, \qquad \ldots(17)$$

$$ikh_x+\frac{dh_y}{dy}+\frac{\partial h_z}{\partial z}=0, \qquad \ldots(18)$$

$$(U-c)\rho+\frac{v}{ik}\frac{d\rho_0}{dy}=0, \qquad \ldots(19)$$

$$iku+\frac{\partial v}{\partial y}+Tv=0. \qquad \ldots(20)$$

Eliminating all variables in favors of v and neglecting the derivatives of $\rho_0(y)$ except the term containing g leads to

$$(U-c)v''+\left[\frac{N^2}{U-c}-U''\right]v+[(U-c)Tv]'-(U-c)k^2 v$$

$$-\frac{S}{(U-c)}(v''-k^2 v)=0, \qquad \ldots(21)$$

where U is the basic velocity in the x-direction which is function of vertical coordinate y, c is the complex wave velocity $v(y)e^{ik(x-ct)}$ is the vertical velocity of the disturbance, k is the wave number, $T=(\log b)'$, $b(y)$ is the width function of the channel $S=\frac{\mu_e H}{4\pi}$ is the magnetic parameter and $N^2=-\frac{\rho' g}{\rho}$. Here primes denote the differentiations with respect to y.

The boundary conditions are given by $v(0)=0=v(D)$, …(22)

Where y=0 denotes the elevation of the deepest point in the channel and y=D denotes the upper surface elevation.

The non-dimensional form of equation (21) can be written as

$$v''+\left[\frac{N^2}{(U-c)^2}-\frac{U''}{(U-c)}\right]v+\frac{1}{(U-c)}[(U-c)Tv]'-k^2v$$

$$-\frac{S}{(U-c)^2}(v''-k^2v)=0. \quad \text{…(23)}$$

For the case of weak applied magnetic field, the term Sv''can be neglected in comparison to the term Sk^2v. Therefore for the weak applied magnetic field the equation (23) reduces to

$$v''+\left[\frac{N^2}{(U-c)^2}-\frac{U''}{(U-c)}\right]v+\frac{1}{(U-c)}[(U-c)Tv]'-k^2v$$

$$+\frac{Sk^2}{(U-c)^2}v=0. \quad \text{…(24)}$$

The associated boundary conditions are

$$v(0)=0=v(1). \quad \text{…(25)}$$

3 Stability Analyses:

Multiplying equation (24) by $\bar{v}$, the complex conjugate of v, integrating the resulting equation over the flow domain and using the boundary conditions (25), we get

$$\int\left(|v'|^2+k^2|v|^2\right)dy-\int\left[\frac{N^2+Sk^2}{(U-c)^2}-\frac{U''}{(U-c)}+\frac{TU'}{(U-c)}\right]|v|^2\,dy$$

$$-\int Tv\bar{v}dy=0. \quad \text{…(26)}$$

The real and imaginary parts of (26) are given by

$$\int\left(|v'|^2+k^2|v|^2\right)dy-\int\left[\frac{(N^2+Sk^2)\left[(U-c_r)^2-c_i^2\right]}{|U-c|^4}-\frac{U''(U-c_r)}{|U-c|^2}+\frac{TU'(U-c_r)}{|U-c|^2}\right]|v|^2\,dy$$

$$-\int T|v'||v|dy \le 0. \qquad \ldots(27)$$

And $c_i \int \left[\frac{(N^2+Sk^2)(U-c_r)}{|U-c|^4} - \frac{U''}{|U-c|^2} + \frac{TU'}{|U-c|^2} \right] |v|^2 dy + \int T|v'||v|dy \ge 0. \qquad \ldots(28)$

Inequality (27) can be written as

$$\int \left(1 - \frac{T}{2k}\right)\left(|v'|^2 + k^2|v|^2\right)dy - \int \left[\frac{(N^2+Sk^2)}{|U-c|^2} - \frac{U''(U-c_r)}{|U-c|^2} + \frac{TU'(U-c_r)}{|U-c|^2} \right] |v|^2 dy$$

$$+ 2c_i^2 \int \frac{(N^2+Sk^2)}{|U-c|^4} |v|^2 dy \le 0. \qquad \ldots(29)$$

Using the transformation $v = (U-c)F$ in the equation (24), we get

$$\left[(U-c)^2 F'\right]' + N^2 F + (TF)' - k^2 (U-c)^2 F + Sk^2 F = 0. \qquad \ldots(30)$$

The corresponding boundary conditions are

$$F(0) = 0 = F(1). \qquad \ldots(31)$$

Multiplying equation (30) by $\overline{F}$, the complex conjugate of F, integrating the resulting equation over the flow domain and using the boundary conditions (2.31), we get

$$\int (U-c)^2 \left[|F'|^2 + k^2|F|^2\right]dy - \int (N^2+Sk^2)|F|^2 dy - \int T'|F|^2 dy - \int TF'\overline{F}dy = 0. \qquad \ldots(32)$$

Separating the real and imaginary parts of expression (32), we get

$$\int \left[(U-c_r)^2 - c_i^2\right]\left[|F'|^2 + k^2|F|^2\right]dy - \int (N^2+Sk^2)|F|^2 dy - \int T'|F|^2 dy$$

$$-\frac{1}{2k}\int T\left[|F'|^2 + k^2|F|^2\right]dy \le 0, \qquad \ldots(33)$$

And $\quad 2c_i \int (U-c_r)\left[|F'|^2 + k^2|F|^2\right]dy - \frac{1}{2k}\int T\left[|F'|^2 + k^2|F|^2\right]dy \le 0. \qquad \ldots(34)$

Using the transformation $v = (U-c)^{\frac{1}{2}} G$ in the equation (24), we get

$$\left[(U-c)G'\right]' - \frac{U''}{2}G - \frac{U'^2}{4(U-c)}G + \frac{N^2+Sk^2}{(U-c)}G - k^2(U-c)G$$

$$+(U-c)(TG)' + \frac{3}{2}TU'G = 0 \qquad \ldots(35)$$

The corresponding boundary conditions are

$$G(0) = 0 = G(1). \qquad \ldots(36)$$

Multiplying equation (35) by $\overline{G}$, the complex conjugate of *G*, integrating the resulting equation over the flow domain and using the boundary conditions (36), we get

$$\int (U-c)\left[|G'|^2 + k^2|G|^2\right]dy - \int \frac{(N^2+Sk^2)}{(U-c)}|G|^2 dy - \int (U-c)T'|G|^2 dy$$

$$-\int (U-c)TG'\overline{G}dy - \frac{3}{2}\int TU'|G|^2 dy + \int \left[\frac{U''}{2} + \frac{U'^2}{4(U-c)}\right]|G|^2 = 0. \qquad \ldots(37)$$

Separating the real and imaginary parts of expression (37), we get

$$\int (U-c_r)\left(1-\frac{T}{2k}\right)\left[|G'|^2 + k^2|G|^2\right]dy - \int \frac{(N^2+Sk^2-\frac{U'^2}{4})(U-c_r)}{|U-c|^2}|G|^2 dy$$

$$-\int T'(U-c_r)|G|^2 dy - \frac{3}{2}\int TU'|G|^2 dy + \int \frac{U''}{2}|G|^2 \le 0. \qquad \ldots(38)$$

$$\int \left(1-\frac{T}{2k}\right)\left[|G'|^2 + k^2|G|^2\right]dy + \int \frac{\left(N^2+Sk^2-\frac{U'^2}{4}\right)}{|U-c|^2}|G|^2 dy - \int T'|G|^2 dy \le 0.$$

$$\ldots(39)$$

From expression (29), we see that if $T \le 0$, $c_r = 0$ and $TU' - U'' \ge 0$ everywhere in the flow domain then for the existence of inequality (29) c_i has to be positive. Thus we have the following theorem:

Theorem 1: If $T \le 0$ and $TU' - U'' \ge 0$ everywhere in the flow domain then the possible non-oscillatory modes are unstable modes.

From expression (29), we see that if $1-\frac{T}{2k} \ge 0$ and $N^2 - UU'' + TUU' \le 0$ everywhere in the flow domain then c_r cannot be zero. Thus we have the following theorem:

Theorem 2: If $1-\frac{T}{2k} \ge 0$ and $N^2 - UU'' + TUU' \le 0$ everywhere in the flow domain then only oscillatory modes exist.

From inequality (34) we see that that if $T \le 0$ everywhere in the flow domain and $c_r = 0$ then for the existence of this inequality c_i has to be negative. Thus we have the following theorem:

Theorem 3: If $T \leq 0$ everywhere in the flow domain then the possible non-oscillatory modes are stable modes.

From inequality (34) we see that for non-oscillatory modes $c_i \leq \frac{T}{2kU}$ at least at one point in the flow domain. Thus we have the following theorem:

Theorem 4: For non-oscillatory modes the growth rate of unstable modes is given by $c_i \leq \frac{T_{\max}}{2kU_{\min}}$.

From inequality (39) we have that if $1-\frac{T}{2k} \geq 0$, $N^2 - \frac{U'^2}{4} \leq 0$, $T' \leq 0$ and $U'' \geq 0$ everywhere in the flow domain then c_r cannot be zero. Thus we have the following theorem:

Theorem 5: If $1-\frac{T}{2k} \geq 0$, $N^2 - \frac{U'^2}{4} \leq 0$, $T' \leq 0$ and $U'' \geq 0$ everywhere in the flow domain then only oscillatory modes exist.

From expression (39) we see that if $1-\frac{T}{2k} \geq 0$, $N^2 - \frac{U'^2}{4} \geq 0$ and $T' \leq 0$ everywhere in the flow domain then c_i has to be zero. Thus we have the following theorem:

Theorem 6: If $1-\frac{T}{2k} \geq 0$, $N^2 - \frac{U'^2}{4} \geq 0$ and $T' \leq 0$ everywhere in the flow domain then only stable modes exist. Thus the conditions mentioned in the above theorem are the sufficient condition for stability.

4 Concluding Remarks:

In this paper, the hydro magnetic stability of stratified shear flows in sea straits of arbitrary cross section has been discussed. Some results related to stability or instability has been obtained for oscillatory and non-oscillatory modes.

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